Dissecting Business Cycles

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October 4, 2023

Abstract

This paper introduces a novel identification strategy to examine the relative role of aggregate supply and demand shocks in driving business cycles. It dissects real GDP fluctuations into long-run and short-run components, identifying them as long-run supply and short-run demand shocks based on conditional correlations of macro variables. Both shocks contribute significantly to business cycle volatility and drive the co-movements of relevant macroeconomic variables. Additionally, the study identifies a second category of long-run supply shocks that do not impact business cycles, revealing substantial normative and policy implications for benchmark DSGE models estimated in a full information setting. By employing theoretical insights and estimation of DSGE models by matching the dynamic causal effects of the identified business cycle shocks, the paper advocates for parameter estimation in a limited information setting.

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1. Introduction

This paper has dual objectives. Firstly, it offers insights into the question of whether there exists a subset of long-run supply shocks that explains significant business cycle volatility and the concurrent co-movement patterns of macroeconomic variables. Essentially, it aims to understand the origins of business cycles by identifying the dynamic causal effects of shocks[1] that explain the business cycle volatility of GDP. The second part of the paper shifts focus to the challenges posed by full-information estimation of dynamic structural general equilibrium models (DSGE) when non-business[2] cycle fluctuations are present in the data-generating process. These models are integral for conducting counterfactual policy analyses, and any bias in parameter estimates can yield significantly divergent normative and policy implications. Consequently, I advocate for estimation within a limited-information setting, utilizing the identified business cycle shocks.

The inquiry into the nature of fluctuations in real gross domestic product (GDP) revolves around the distinction between transitory and persistent disturbances. A central question pertains to the characterization of recessions: do they entail short-run deviations of GDP from its long-run trend, or do they predominantly signify shifts in the underlying trend itself? The literature on business cycles grapples with this fundamental query, delving into the relative role of real determinants such as productivity in long-run equilibrium versus the short-run fluctuations of demand that steer business cycle dynamics. Discerning whether recessions stem from long-run supply or short-run demand shocks holds significant implications for formulating effective policy responses by central banks. In contrast to the existing literature, this paper presents evidence supporting the significant roles of both long-term supply shocks and short-term demand shocks in driving business cycles.

While this paper emphasizes the significance of both long-term supply and short-term demand shocks as sources of business cycles, it's essential to note that the identification methodology remains agnostic in categorizing these shocks as supply or demand-related. Instead, its primary objective is to identify long-run and short-run shocks relevant to explaining business cycle fluctuations in macroeconomic variables, such as GDP. Consequently, this approach also allows for the evaluation of the impact of short-run supply shocks, such as cost-push shocks, on business cycles.

[1] Going forward referred as business-cycle shocks

[2] The shocks that don't explain significant business cycle volatility of GDP

The labeling of shocks as 'supply' or 'demand' is based on conditional correlations observed among macro variables in the vector autoregression (VAR) and aligns with the standard new Keynesian framework.

The conventional framework of Real Business Cycle (RBC) models and their extensions positions Total Factor Productivity (TFP) shocks as pivotal drivers of labor wedges and consequent business cycle patterns. Nevertheless, a series of empirical investigations[3], has contested this hypothesis. They argue against the notion by highlighting the estimated conditional correlations of hours and productivity display either zero or negative associations for short-run and long-run technology shocks. This discordance has prompted a reevaluation of demand shocks as plausible contributors to business cycle dynamics, particularly in light of their compatibility with a subset of New Keynesian models. Empirical findings from the studies based on structural vector autoregressions[4] (SVAR) have further reinforced the case for demand-driven business cycles.

This paper uses a novel extension of SVAR methodology to reevaluate the respective roles of aggregate supply and demand shocks in driving business cycles. Previous studies in the SVAR literature assumed the existence of only one category of long-run productivity shocks that may or may not drive business cycles. In contrast, this paper employs an empirical approach that allows for the consideration of two categories of long-run supply shocks; one that drives business cycles and the other one that does not. Specifically, it dissects business cycle fluctuations into their constituent long-run (low-frequency) and short-run (high-frequency) components. This empirical strategy builds on the contributions of Uhlig (2003) and Angeletos et al. (2020), aimed at identifying the set of shocks that account for the maximum business cycle volatility observed in real GDP. However, this paper in order to identify two business cycle shocks restricts one of these shocks from explaining low-frequency fluctuations in either real GDP, real consumption, TFP, or labor productivity. This methodology allows us to ask the question of whether there exists a subset of long-run shocks that drive business cycles.

The classification of the two orthogonal shocks, identified as business cycle (BC) supply and demand shocks, relies on examining the impulse response functions (IRFs) and business cycle volatility of the remaining variables within the VAR model. Both of these shocks independently contribute to the co-movement observed in busi-

[3]See Gali (1999), **?**, & Angeletos et al. (2020)

[4]See Blanchard and Quah (1989), Angeletos et al. (2020) & Benhima and Poilly (2021)

ness cycles and account for a major share of the volatility exhibited in output, consumption, investment, hours worked, and labor productivity. The short-run shock, subject to the restriction of not explaining long-run fluctuations, is identified as a 'BC-demand' shock, as it leads to significant procyclical fluctuations in real GDP, inflation, and federal funds rates. Nevertheless, the unrestricted shock also results in long-run impacts on the key macroeconomic quantity variables, along with TFP. This long-run shock is identified as a 'BC-supply' shock, characterized by long-run IRFs of output, consumption, investment, TFP, and labor productivity, and giving rise to countercyclical inflation.

The two identified business cycle shocks combined explain $\approx 99\%$ business cycle volatility of GDP, but $\approx 51\%$ long-run volatility of GDP. Drawing upon the long-run volatility of macro variables explained, this study also provides evidence that a significant portion of long-run shocks does not exert an influence on business cycles (henceforth, long-run non-business cycle shocks). Based on the evidence of such non-business cycle fluctuations in the data, the second part of the paper focuses on the challenge of parameter estimation in medium-scale DSGE models when estimated under full information setting. I formally argue that in the presence of cross-frequency restrictions within DSGE models and a notable fraction of long-run non-business cycle shocks in the data, parameter estimates become biased, with consequential implications for these models' business cycle outcomes. Central banks often employ various versions of these models estimated in full information settings to analyze the role of economic frictions and for policy analysis. Instead, this paper argues for estimation in a limited information setting i.e. information relevant to business cycle fluctuations.

To unpack the normative and policy implications of models estimated under full information, this study utilizes the benchmark medium-scale DSGE model proposed by Smets and Wouters (2007), hereafter referred to as SW07, originally estimated using Bayesian likelihood estimation and estimates it using impulse response matching methodology akin to Christiano et al. (2005). This model includes a range of nominal and real frictions, along with seven structural shocks. Studying their model serves two primary objectives. First, it offers a more comprehensive structural interpretation of the 'demand' shock identified in the empirical analysis, as the model can accommodate shocks that generate both standard business-cycle dynamics and deviations from those patterns. Notably, the model's setup underscores the adaptability and applicability of the proposed identification strategy and results across various

VAR specifications, given that SW07 employs a seven-variable VAR of observables without incorporating TFP as an observable.

Second, conventional frictionless RBC models typically predict that long-run TFP fluctuations have an expansionary effect, while other macroeconomic models, which account for sticky prices and imperfect information, suggest the opposite in the short run. These alternative models propose that technological advancements may initially lead to short-run declines in employment or hours worked due to price rigidities but result in long-run increases when prices can adjust. This dynamic leads to a positive conditional correlation between the output gap and inflation, even in response to long-run TFP shocks, resulting in divine coincidence akin to demand-driven business cycles in new Keynesian framework.

However, when the SW07 model is estimated using impulse response matching, it reveals a policy tradeoff for monetary authorities due to a negative comovement between inflation and the output gap. This contrast in results challenges the normative and policy implications compared to the same model estimated under a full information setting i.e. Bayesian likelihood estimation, highlighting the sensitivity of policy recommendations to the choice of estimation methodology.

This paper makes contributions to two distinct strands of literature within the macroeconomic literature on business cycles. The first strand, rooted in the SVAR framework[5] revolves around quantifying the relative impact of aggregate supply and demand shocks on the fluctuations inherent in business cycles. However, the analyses conducted in these aforementioned studies[6] collectively challenge the conventional notion of long-run TFP fluctuations as the primary driver of business cycles. Instead, these investigations posit demand shocks as the pivotal drivers behind business cycle dynamics.

In contrast, this study deviates from this perspective and aligns itself with the SVAR evidence presented by Beaudry and Portier (2006), Chahrour and Jurado (2018) & Chahrour et al. (2023). These works advocate for long-run TFP-driven business cycles while not explicitly addressing the relative influence of demand shocks. Building upon this foundation, the current paper advances the comprehension of the role played by long-run supply shocks in shaping business cycles. It achieves this by employing the benchmark VAR model rooted in the framework of Angeletos

[5]See Blanchard and Quah (1989), Shapiro and Watson (1998), & Angeletos et al. (2020)

[6]See Gali (1999), **?**, Barsky et al. (2014), Barsky and Sims (2011), Neville et al. (2014a), Kurmann and Sims (2021), Benhima and Poilly (2021)

et al. (2020), but dissects business cycle fluctuations into their distinct long-run supply and short-run demand components. This approach highlights the intricate dynamics underpinning business cycles, thus accentuating the significant contributions of both long-run supply shocks and demand shocks in driving patterns of co-movement.

The second strand of literature to which this paper contributes is the literature on the identification of macroeconomic equations through structural shocks. Notable works in this domain include Rotemberg and Woodford (1997) and Christiano et al. (2005), which estimate DSGE models by matching impulse response functions. More recently, Barnichon and Mesters (2020) introduced a new approach involving regressions in impulse response space, and Lewis and Mertens (2022) presented an improved approach. This paper extends this literature by advocating for the use of conditional variation in identified business cycle shocks to discipline structural model parameters, as opposed to the Bayesian likelihood approach using unconditional moments.

The paper is organized as follows. Section 2 outlines the data and methodology employed to dissect business cycle fluctuations and identify business-cycle (BC) supply and demand shocks. In Section 3, the primary empirical findings are presented. Building on the evidence of long-run non-business cycle shocks from the previous section, Section 4 formally discusses the issue of biased estimation in a full information setting. Moving on to Section 5, I estimate the parameters of the medium-scale Dynamic Stochastic General Equilibrium (DSGE) model of SW07 required to match the identified dynamic causal effects of business cycle shocks. This section also undertakes a comparative analysis of the model, estimated using conditional moments, and the Bayesian likelihood estimation of the original estimated model. Finally, Section 6 concludes.

2. Data and Method

The data utilized in the main analysis of this study includes quarterly observations of ten macroeconomic variables. These variables include the unemployment rate (u), the real per capita levels of GDP (Y), hours worked (h), investment (I), consumption (C), labor productivity in the nonfarm business sector (Y/h), the level of utilization-adjusted total factor productivity (TFP), the labor share (wh/Y), the inflation rate (π), as measured by the rate of change in the GDP deflator, and the nominal interest rate (R), as measured by the federal funds rate. The sample for this study begins in 1955:I, the earliest date of availability for the federal funds rate, and ends in 2019:IV.

The Vector Autoregression (VAR) model employed in this study takes the form:

$$A(L)Y_t = \mu_t, \tag{1}$$

Where Y_t is a vector of n macroeconomic variables under examination, $A(L)$ is a matrix polynomial represented by the sum of $A_\tau L^\tau$, with $A(0) = A_0 = I$, and l is the number of lags included in the VAR. The vector of residuals, μ_t, follows the Assumption of $E(\mu_t \mu_t') = \Sigma$ for a positive definite matrix Σ. The large size of the VAR necessitated the use of Bayesian methods and a Minnesota prior for estimation. The baseline specification employed 2 lags, as suggested by standard Bayesian criteria.

The method is based on the Assumption that there exists a linear relationship between the residuals, denoted by μ_t, and a set of mutually orthogonal shocks, represented by ε_t. Mathematically, this relationship can be represented by the equation $\mu_t = C\varepsilon_t$, where C is an invertible $n \times n$ matrix.

Another key Assumption in the analysis is that the othogonal shocks, ε_t, are independently and identically distributed over time. Additionally, we assume that the covariance matrix of these shocks is equal to the identity matrix, I. The interpretation of these orthogonal shocks as "structural" shocks, such as exogenous changes in supply or demand, will be based on the impulse response functions (IRFs) of the variables included in the VAR. By examining the dynamic responses of the variables to a shock in a set of variable, we can gain insight into the underlying causes of the fluctuations in the data.

To identify these shocks, the matrix C is decomposed into the Cholesky decomposition of the VAR residuals covariance matrix, $\tilde{C}$, and an orthonormal matrix, Q. This leads to the relationship $\varepsilon_t = C^{-1}\mu_t = Q'\tilde{C}^{-1}\mu_t$, where each column of Q corresponds to a shock in ε_t.

However, simply satisfying $QQ' = I$ and $CC' = \Sigma$ does not suffice for identifying the underlying shocks. To do so, we impose additional restrictions on Q based on the requirement that it contains the maximal share of all the information in the data about the volatility of a specific variable in a specific frequency band. This approach is different from the typical SVAR exercises in the literature which employ exclusion or sign restrictions motivated by specific theories.

The Wold representation of the VAR model is given by the following equation:

$$Y_t = B(L)\mu_t, \tag{2}$$

Where $B(L)$ is an infinite matrix polynomial, and μ_t represents the residuals. We then substitute $\mu_t = \tilde{C}Q\varepsilon_t$, where $\tilde{C}$ is the Cholesky decomposition of the VAR residuals covariance matrix, and Q is an orthonormal matrix, leading to the following representation:

$$Y_t = D(L)Q\varepsilon_t = \Theta(L)\varepsilon_t, \tag{3}$$

Where $D(L)$ and $\Theta(L)$ are infinite matrix polynomials, with $D_\tau \equiv B_\tau\tilde{C}$ and Θ_τVD$_\tau Q$ for all $\tau \in 0, 1, 2, \ldots$. The sequence $\{\Theta_\tau\}_{\tau=0}^{\infty}$ represents the impulse response functions (IRFs) of the variables to the structural shocks.

As mentioned, to interpretation of the structural shocks is based on the dynamic responses of the variables to the respective shock. By considering the (i, j) element of the matrix Θ_τ , one may identify the effect of the jth shock on the ith variable at horizon τ. This allows to gain insight into the underlying causes of the fluctuations in the data, and identify which shocks are likely to represent exogenous changes in supply or demand.

2.1 Identification Strategy

The identification strategy used in this paper builds upon the “max-share" approaches first introduced in the literature by Faust (1998) and Uhlig (2003). These approaches have been subsequently adapted and expanded upon by several other authors, including Barsky and Sims (2011), Kurmann and Otrok (2013), Neville et al. (2014b), Angeletos et al. (2020), Kurmann and Sims (2021) and Charhour et al. (2021) among others. The implementation of this strategy is in the frequency domain, similar to that used by Angeletos et al. (2020), hereafter referred as ACD, where the goal is to identify a reduced-form shock that explains the maximum volatility of a targeted variable in a specific frequency band.

In contrast to the ACD approach, this paper adopts a distinct perspective by emphasizing the role of two specific shocks as pivotal drivers of business cycle fluctuations. This standpoint is underpinned by the scree plot illustrated in Figure 1, where the x-axis is limited to 10 in accordance with the maximum number of eigenvalues of a 10-variable VAR. The observed trend in the eigenvalues reveals a convergence to zero from the third eigenvalue onward, while the first two principal components retain significant values. ACD, on the other hand, focuses the first eigenvalue, explaining 80% of business cycle volatility in output, as the main business cycle shock. Drawing from this empirical evidence, the argument advanced in this paper asserts the necessity of two orthogonal business cycle shocks to effectively account for the fluctuations in real GDP per capita across business cycles.

The key novel contribution of this paper lies in the dual employment of targeting and constraining methodologies to identify the two orthogonal business cycle shocks and better explain the volatility of specific variables over a certain frequency band while constraining them to fluctuations of another variable in the same or a different frequency domain.

To illustrate this further, I use a standard data generating process (DGP), where the vector of structural shocks ε_t is decomposed into two distinct categories: business cycle shocks and non-business cycle shocks.

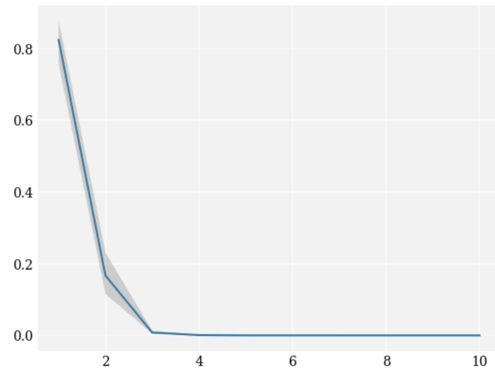
$$\epsilon_t' = [\quad \underbrace{\epsilon_{B,t}^{sr} \qquad \epsilon_{B,t}^{lr}}_{\text{Business cycle shocks}} \quad \underbrace{\epsilon_{NB,t}^{long-run} \qquad \epsilon_{NB,t}^{residual}}_{\text{Non-Business Cycle shocks}} \,]$$

The business cycle shocks are represented as $\epsilon_{B,t}^{sr}$ and $\epsilon_{B,t}^{lr}$, while the non-business cycle shocks are represented as $\epsilon_{NB,t}^{long-run}$ and $\epsilon_{NB,t}^{residual}$. The subscript 'B' in $\epsilon_{B,t}^{sr}$ and $\epsilon_{B,t}^{lr}$ denotes that these orthogonal shocks are identified by maximizing their contribution to the volatility of the targeted variable, real GDP per capita, over the business cycle frequency band, or frequencies pertaining to a time period of 6-32 quarters.

The superscript 'sr' in $\epsilon_{B,t}^{sr}$ denotes a restriction of the shock to explain long-run volatility of real GDP per capita at long-run frequencies, i.e. frequency bands pertaining to time periods of 20-100 years. In other words, $\epsilon_{B,t}^{sr}$ is identified by simultaneously targeting the same variable and frequency bands as $\epsilon_{B,t}^{lr}$, but with the additional restriction of explaining long-run volatility of real GDP per capita. The results are robust if the restriction is applied to long-run fluctuations of consumption, TFP or labor productivity instead of GDP.

On the other hand, non-business cycle shocks are further classified into two subcategories: long-run shocks and residual shocks. The long-run shocks, identified as $\epsilon_{NB,t}^{long-run}$, are orthogonal to the business cycle shocks and lead to persistent changes in real GDP per capita or fluctuations pertaining to frequency bands with time periods above 20 years. These shocks explain long-run fluctuations of GDP but not the business cycle fluctuations. On the other hand, the residual shocks, identified

Figure 1: Scree Plot



Eigenvalues for a spectral matrix of GDP at business cycle frequency band. Horizontal axis: Total principal components or eigenvalues.

as $\epsilon_{NB,t}^{residual}$, are orthogonal to both the business cycle shocks and the long-run non-Business cycle shocks. These residual shocks capture all other non-business cycle shocks that are not captured by the other two categories.

The objective of this paper is to partially identify the business cycle shocks, and this further classification of ϵ_t allows for a more detailed examination of the results. This classification of structural shocks is useful for highlighting the results of our analysis in comparison to existing literature, such as the works of Blanchard and Quah (1989) and Angeletos et al. (2020).The inclusion of non-business cycle shocks introduces model misspecification, impacting both the SVAR approach and the local identification analysis of linearized DSGE models in full-information contexts. Sections 4 & 5 of the paper elucidates this phenomenon using a SW07 framework.

In the Wold representation from the previous subsection, the variable Y_t can be represented as:

$$Y_t = D(L)Q\varepsilon_t$$

where ε_t is a white noise process and Q is an orthonormal matrix. The spectral density of a variable y_j in Y_t in the frequency band $[\underline{f}, \bar{f}]$ can be represented by $\mathcal{D}(y_j, \underline{f}, \bar{f})$:

$$\mathcal{D}(y_j, \underline{f}, \bar{f}) = \int_{\underline{f}}^{\bar{f}} \left(\overline{D^j\left(e^{-if}\right)} D^j\left(e^{-if}\right) \right) df$$

where the sequence $\{D_\tau\}_{\tau=0}^{\infty}$ represents the Cholesky transformation of the VAR residuals, and D_τ^j represents the jth row of the matrix D_τ.

To identify a shock $\epsilon_{1,t}$, we need to find the column of the orthonormal matrix Q that represents the shock and explains the maximum volatility of y_j in the frequency band $[\underline{f}, \bar{f}]$. This can be represented as:

$$q_1 \equiv \arg\max_q \int_{\underline{f}}^{\bar{f}} \left(\overline{D^j\left(e^{-if}\right)q} D^j\left(e^{-if}\right)q\right) df \tag{4}$$

$$\equiv \arg\max_q q^{'} \mathcal{D}(y_j, \underline{f}, \bar{f}) q \tag{5}$$

$$\text{s.t. } q'q = 1, \tag{6}$$

Similarly, if we need to identify a shock $\epsilon_{2,t}$ that explains the maximum volatility of y_j in the frequency band $[\underline{f}, \bar{f}]$, but not the volatility of y_k in the frequency band $[\underline{\omega}, \bar{\omega}]$. This can be represented as:

$$q_2 \equiv \arg\max_q \int_{\underline{f}}^{\bar{f}} \left(\overline{D^j\left(e^{-if}\right)q} D^j\left(e^{-if}\right)q\right) df - \int_{\underline{\omega}}^{\bar{\omega}} \left(\overline{D^k\left(e^{-i\omega}\right)q} D^k\left(e^{-i\omega}\right)q\right) d\omega \tag{7}$$

$$\equiv \arg\max_q q^{'} \mathcal{D}(y_i, \underline{f}, \bar{f}) q - q^{'} \mathcal{D}(y_k, \underline{\omega}, \bar{\omega}) q \tag{8}$$

$$\text{s.t. } q'q = 1, \tag{9}$$

Building on this, the objective is to identify two orthogonal shocks: $q_{B,t}^{sr}$ and $q_{B,t}^{lr}$. These shocks simultaneously should explain the volatility of real GDP per capita at business cycle frequency, but restricting $q_{B,t}^{sr}$ from explaining the long-run volatility of GDP. The objective function is as follows:

$$q_B^{sr}, q_B^{lr} \equiv \arg\max_{q_B^{lr}, q_B^{sr}} q_B^{lr\prime} \mathcal{D}\left(GDP, \frac{2\pi}{32}, \frac{2\pi}{6}\right) q_B^{lr} + q_B^{sr\prime}\left(\mathcal{D}\left(GDP, \frac{2\pi}{32}, \frac{2\pi}{6}\right) - \mathcal{D}\left(GDP, \frac{2\pi}{400}, \frac{2\pi}{80}\right)\right) q_B^{sr} \tag{10}$$

$$\text{s.t. } q_B^{lr\prime} q_B^{lr} = 1, q_B^{sr\prime} q_B^{sr} = 1, q_B^{lr\prime} q_B^{sr} = 0 \tag{11}$$

To examine business cycle fluctuations, the framework follows Stock and Watson (1999) where the business cycle frequency band is defined as $\left[\frac{2\pi}{32}, \frac{2\pi}{6}\right]$, while the long-run frequency band is specified as $\left[\frac{2\pi}{400}, \frac{2\pi}{80}\right]$. The upper bound of the long-run frequency band is based on the findings of ACD, while the lower bound of $\frac{2\pi}{400}$ (100 years or 400 quarters) instead of ≈ 0 is chosen to avoid any potential non-stationarity issues in the estimation process.

Additionally, the optimization procedure is subject to the constraint $q_B^{lr\prime} q_B^{sr} = 0$, ensuring that the long-run and short-run frequency bands are orthonormal. This means that the inner product of the two vectors is zero and each vector has a unit length. The resulting shocks, q_B^{lr} and q_B^{sr}, are interpreted as supply and demand shocks, respectively, based on their respective impulse response functions.

2.2 Solution Method

The problem of identification, expressed in equation 10, can be represented as,

$$\max_{X \in \mathbb{R}^{n \times p}} \mathcal{F}(X), \quad \text{s.t. } X^\top X = I_p \tag{12}$$

where,

$$\mathcal{F}(X) = \sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A}_i \mathbf{x}_i \tag{13}$$

A maximization of the sum of quadratic forms generated by different matrices. The objective is to find the set of orthonormal elements in $\mathbb{R}^n$ (where $k \leqslant n$) that maximizes the functional $\sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A}_i \mathbf{x}_i$, subject to the constraint $X^\top X = I_k$.

This problem is distinct from a standard principal component analysis, where the objective is to to find a system of k orthonormal elements in $\mathbb{R}^n$ (where $k \leqslant n$) that maximize the functional $\sum_{i=1}^{k} \mathbf{x}_i^T \mathbf{A} \mathbf{x_i}$, it is known that the k largest eigenvalues of matrix $\mathbf{A}$ and their associated orthonormal eigenvectors are the solution to this optimization problem. These eigenvectors, or principal components, represent the most informative directions in the data and capture the maximum amount of variation in the data.

While in equation 12 the objective is to find the maximum of sum of quadratic forms generated by different matrices. In most cases it cannot be the sum of the largest eigenvalues of corresponding matrices, because the eigenvectors corresponding to the maximal eigenvalues of $\mathbf{A}_i$'s are usually not pairwise orthogonal.

In Bolla and Ziermann (1998), the authors prove the existence and uniqueness of solutions for optimization problems of this nature. The solution is determined using an adaptive feasible Barzilai-Borwein-like (AFBB) algorithm. The global convergence of this algorithm is demonstrated in Jiang and Dai (2014), through the use of an adaptive nonmonotone line search.

3. Empirical Results

This section presents the main empirical findings and discusses a few tentative lessons for theory.

Figure 2 illustrates the impulse response functions (IRFs) of all variables to the $q^{lr}_{B,t}$ shock. This shock induces co-movement among key business cycle variables and plays a substantial role in explaining business cycle volatility, accounting for over 50% of the volatility in real per capita GDP. Additionally, the TFP impulse response to the same shock exhibits remarkable persistence, explaining approximately 54% of its long-run volatility. Furthermore, the shock is responsible for significant fluctuations in consumption and investment, both at business-cycle frequencies and in the long run.

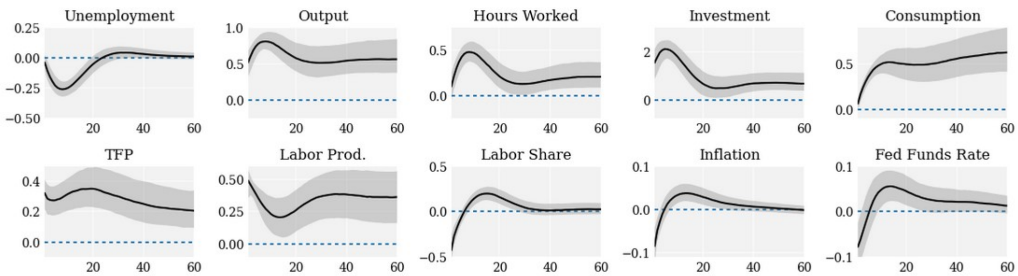
In the context of conditional correlations involving real GDP, TFP, inflation, and interest rates, the shock has been identified as a long-run TFP shock within a new Keynesian model. In accordance with a common Assumption in the business cycle literature, long-run TFP changes are considered exogenous. With the presence of nominal rigidities, a positive TFP shock leads to a decrease in inflation due to a decline in marginal cost, coupled with a decrease in interest rates following a standard inflation targeting monetary rule.

Figure 3 reports the impulse response functions (IRFs) of all the variables to the $q^{sr}_{B,t}$ shock. The shock results in co-movement of the key business cycle variables and explains significant business cycle volatility but not long-run. The impulse responses are quite significant on impact. No movement or business cycle volatility explained for TFP.

In contrast to the previous shock, this particular one exhibits no short-run or long-run movements in TFP or labor-productivity IRFs. Furthermore, the positive comovement observed in TFP, consumption, inflation, and interest rates allows us to identify this shock as a demand shock. While the previous supply shock was characterized as a long-run TFP shock, the precise interpretation of this structural shock is model-dependent. In the forthcoming section, I will present arguments, drawing from SW07, medium-scale DSGE model, to support the notion that the identified shock represents a risk-premia shock. However, it is essential to acknowledge that the further structural interpretation of the demand shock varies depending on the perspective of the model utilized.

The main business cycle shock within the ACD framework is essentially a linear combination of the above two shocks. This MBC shock accounts for 26% of long-run TFP volatility, in contrast to the 53% attributed to the long-term supply shock and 17% for the business cycle fluctuations in the federal funds rate, compared to the 52.5% explained by the identified short-term demand shock. Section 7.2 in the appendix offers a bivariate example that elucidates why this MBC shock is likely to be a linear combination of the identified business cycle shocks. The example explains how this combination can result in higher volatility for the targeted variable while reducing volatility for variables moving in opposite directions due to the two orthogonal shocks.

Figure 2: Impulse Response Functions to Supply Shock



Impulse Response Functions of all the variables to the identified long-run supply shock. Horizontal axis: time horizon in quarters. Shaded area: 68 percent Highest Posterior Density Interval (HPDI).

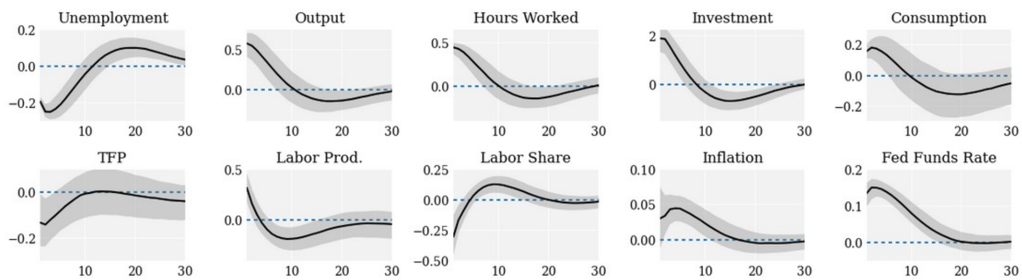
Table 1: Supply Shock, Variance Contributions

	u	Y	h	I	C
Short run (6–32 quarters)	32	53.1	29.8	40.6	32.7
	[21.2,43.8]	[33.4,70.5]	[21.2,40.2]	[25.7,57.1]	[25.6,39.9]
Long run (80–400 quarters)	35.7	51.8	20.6	47.9	51.4
	[19.1,55.5]	[26.7,72.7]	[4.8,48.2]	[22.4,70.2]	[26.1,71.9]

	TFP	Y/h	wh/Y	Δp	FFR
Short run (6–32 quarters)	13.7	36.6	30.3	19	19.3
	[6.5,23.7]	[26.4,45.2]	[15.2,42.9]	[10.7,29.2]	[9.2,36]
Long run (80–400 quarters)	53.7	54.3	41.9	14.2	15.9
	[29.4,71.4]	[29,72.1]	[18,63]	[5.8,29.9]	[6.4,34.7]

Notes: Variance contributions of the identified long-run supply shock at two frequency bands. The first row (Short run) corresponds to the range between 6 and 32 quarters, the second row (Long run) to the range between 80 quarters and ∞. The notation used for the variables is the same as that introduced in Section I. 68 percent HPDI in brackets.

Figure 3: Impulse Response Functions to Demand shock



Impulse Response Functions of all the variables to the identified short-run demand shock. Horizontal axis: time horizon in quarters. Shaded area: 68 percent Highest Posterior Density Interval (HPDI).

Table 2: Demand Shock, Variance Contributions

	u	Y	h	I	C
Short run (6–32 quarters)	48.6	45.8	40.9	44.2	23
	[36.8,59.2]	[28.4,65.3]	[29.9,49.2]	[27.9,59.5]	[16.3,31.3]
Long run (80–400 quarters)	5.1	0.2	2.3	0.4	0.15
	[1.8,13.4]	[0.03, 1]	[0.5, 8.8]	[0.08, 1.9]	[0.02, 0.8]
	TFP	Y/h	wh/Y	Δp	FFR
Short run (6–32 quarters)	7.4	22.5	16.5	10.9	50.5
	[2.5,15.6]	[12.5,33.9]	[7,32.1]	[5.6,18.8]	[36.6,60.7]
Long run (80–400 quarters)	0.04	0.04	1.1	3.7	12.9
	[0, 0.2]	[0.01, 0.2]	[0.2, 4]	[1.1, 9.7]	[4.6,26.3]

Notes: Variance contributions of the identified short-run shock at two frequency bands. The first row (Short run) corresponds to the range between 6 and 32 quarters, the second row (Long run) to the range between 80 quarters and ∞. The notation used for the variables is the same as that introduced in Section I. 68 percent HPDI in brackets.

4. Full Information Estimation: Challenges

In this section, the aim is to illustrate how the Bayesian likelihood estimation poses challenges due to the presence of long-run shocks that don't result in business cycles. This would involve representing the Gaussian log-likelihood function of the state-space models in the frequency domain. The frequency domain provides the decomposition of fluctuations of a variable into fluctuations of different periodicity. This breakdown helps us separate long-term and short-term fluctuations of the variable, which is important for this paper's purpose.

Let's start with a canonical representation of a linearized DSGE model:

$$\Gamma_0 S_t = \Gamma_1 S_{t-1} + \Psi\epsilon_t + \Pi\eta_t \tag{14}$$

Where 1) S_t is a vector of model variables that include (i) the endogenous variables, (ii) the conditional expectations, (iii) the variables from exogenous processes if they are serially correlated; 2) ε_t is a vector of exogenous disturbances; 3) η_t is a vector of expectation errors satisfying $E_{t-1}\eta_t = 0$ for all t; 4) Γ_0, Γ_1 and Π are coefficient matrices; 5) Ψ a diagonal matrix with standard deviations of the exogenous disturbances.

Assuming the above set of equilibrium conditions that represent optimality conditions have a state-space representation and mapping to a vector of observables $\mathbf{Y}_t$:

$$S_t = \Theta_1(\theta)S_{t-1} + \Theta_\epsilon(\theta)\Psi(\theta_1)\epsilon_t \tag{15}$$

$$Y_t = A(L)S_t = A(L)\left(I - \Theta_1(\theta)L\right)^{-1}\Theta_\epsilon(\theta)\Psi(\theta_1)\epsilon_t = \mathbf{D}(L;\theta)\mathbf{\Theta}_\epsilon(\theta)\mathbf{\Psi}(\theta_1)\epsilon_t \tag{16}$$

Where 1) θ_1 is a vector of standard deviations of exogenous shocks; 2) θ is a vector of all deep parameters of the DSGE model except the ones in θ_1.

The model implied Spectral Density of variable **k** in Y_t due to shock **l** in ε_t at frequency ω is represented as:

$$\mathcal{SD}(\omega, k, l; \theta, \theta_1) = \frac{1}{2\pi}\mathcal{M}(\omega, y_k, l; \theta)\sigma_l^2, \quad \text{where} \quad \mathcal{M}(\omega, y_k, l; \theta) = \left|\mathbf{D}^k(e^{i\omega};\theta)\mathbf{\Theta}_\epsilon^l(\theta)\right|^2 \tag{17}$$

For the sake of tractability, let's make a simplifying Assumption. Let's assume the true data generating process of the variable y_k involves two exogenous shocks ϵ_B & ϵ_{lNB}. Here, ϵ_B represents a business cycle shock, while ϵ_{lNB} represents a long-run that doesn't cause business cycles.

Since standard DSGE models don't allow both categories of long-run shocks by having restrictions that allow one to cause business cycles but not the other, the exogenous shock process vector ε in the canonical representation above comprises only a business cycle shock[7], which is represented as ϵ_B with a variance of $\sigma^2_{\mathbf{B}}$.

The log-likelihood function of the above state space model in the frequency domain following Harvey (1989) is as follows:

$$\log L\left(\theta, \theta_1\right) = -\sum_{j=1}^{T}\left(\log \frac{1}{2\pi}\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)\sigma^2_{\mathbf{B}} + \frac{I(\omega_j, y_k)}{\frac{1}{2\pi}\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)\sigma^2_{\mathbf{B}}}\right) \tag{18}$$

where, $\omega_j = \frac{2\pi t}{T}$. The likelihood function depends on two arguments: the spectral density of the model $\mathcal{SD}(\omega_j, k, l; \theta, \theta_1)$ and the periodogram $I_y\left(\omega_j, k\right)$ which is the data implied volatility at frequency ω_j,

$$I\left(\omega_j, k\right) = \frac{1}{2\pi}\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma^2_{\mathbf{B}} + \frac{1}{2\pi}\mathcal{D}(\omega_j, y_k, \mathbf{lNB})\sigma^2_{\mathbf{lNB}} \tag{19}$$

where as defined in section 2.1, $\mathcal{D}(\omega_j, y_k, l) = q_l^{'}\overline{D^k\left(e^{-i\omega_j}\right)}D^k\left(e^{-i\omega_j}\right)q_l$

Maximising $\log$ L with respect to $\sigma^2_{\mathbf{B}}$ gives:

$$\tilde{\sigma}^2_{\mathbf{B}}(\theta) = \frac{2\pi}{T}\sum_{j=1}^{T}\frac{I(\omega_j, y_k)}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)} = \frac{2\pi}{T}S(\theta) \tag{20}$$

Following, Harvey (1989) (pg. 193), the exogenous shock variance may therefore, be concentrated out of the likelihood function, with the result that maximizing $\log L$ in (18) is equivalent to minimizing $S(\theta)$, where

$$S(\theta) = \underbrace{\sum_{j=1}^{t}\frac{I(\omega_j, y_k)}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{long-run volatility}} + \underbrace{\sum_{j=t+1}^{T}\frac{I(\omega_j, y_k)}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{short-run volatility}} \tag{21}$$

[7]Subsection 7.4 shows how multiple shocks in the model can be mapped to a business cycle shock.

$$S(\theta) = \underbrace{\sum_{j=1}^{t} \frac{1}{2\pi} \frac{\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma^2_{\mathbf{B}} + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\sigma^2_{\mathbf{lNB}}}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{long-run volatility}} + \underbrace{\sum_{j=t+1}^{T} \frac{1}{2\pi} \frac{\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma^2_{\mathbf{B}} + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\sigma^2_{\mathbf{lNB}}}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{short-run volatility}} \tag{22}$$

DSGE models estimated in the time domain are equivalent to fitting the model over the whole spectral density. These models generate cross-frequency restrictions, the presence of information in the estimation that the model is not intended to explain may affect the estimates. The area under the power spectrum over the range $[-\pi, \pi]$ is equal to the variance of the process (Harvey (1989), pg. 58). More generally,

$$\underbrace{\sum_{j=1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}_{\text{long-run}} + \underbrace{\sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}_{\text{short-run}} = 1 \tag{23}$$

Thus, the power spectrum may be viewed as a decomposition of the variance of the process in terms of frequency. In summary, the estimation of vector θ of model parameters is equivalent to minimizing $S(\theta)$ in (22) subject to the constraint (23).

To understand the bias introduced by the presence of long-run Non-business cycle fluctuations $\left(\sigma^2_{\mathbf{lNB}} > 0\right)$ in the data $(I_y(\omega_j, k))$ using Bayesian likelihood estimation, we make the following three Assumptions,

Assumption 1 *: Suppose* $\exists\, \theta^*$ *s.t.* $\mathcal{D}(\omega_j, y_k, \boldsymbol{B}) = \mathcal{M}(\omega_j, y_k, \boldsymbol{B}; \theta^*)\ \forall\, \omega_j$ *where* $j \in \{1, 2, \cdots, T\}$

Assumption 1 implies that, the above DSGE model is well-specified for business cycle fluctuations. This implies that there exists a vector of parameters (θ^*) such that model implied volatility due to business cycle shock $(\mathcal{SD}(\omega_j, y_k, \mathbf{B}; \theta^*))$ is equal to data implied volatility of the business cycle shock $\left(\frac{1}{2\pi}\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma^2_{\mathbf{B}}\right)$ at all frequencies ω_j.

Assumption 2 *: Suppose* $\exists\, \theta^{'}$ *s.t.* $\frac{\mathcal{D}(\omega_j, y_k, \boldsymbol{B}) + \mathcal{D}(\omega_j, y_k, \boldsymbol{lNB})\kappa}{\mathcal{D}(y_k, \boldsymbol{B}, \boldsymbol{lNB})} = \mathcal{M}(\omega_j, y_k, \boldsymbol{B}; \theta^{'})\ \forall\, \omega_j$ *where* $j \in \{1, 2, \cdots, T\}$

Assumption 2 implies that there exists a vector of parameters $(\theta^{'})$ such that model implied volatility due to a business cycle shock $\left(\mathcal{SD}(\omega_j, y_k, \mathbf{B}; \theta^{'})\right)$ is equal to data implied normalized volatility of both a business cycle and non-business cycle shock $\left(\sum_{j=1}^{t} \frac{1}{2\pi}\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma^2_{\mathbf{B}} + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\sigma^2_{\mathbf{lNB}}\right)$ at all frequencies ω_j.

where, $\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB}) = \prod_{j=1}^{T} \left(\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa\right)$ & $\kappa = \frac{\sigma^2_{\mathbf{lNB}}}{\sigma^2_{\mathbf{B}}}$

$\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})$ in Assumption 2 is used for normalization of spectral density due to both business cycles and non-business cycle shocks such that the combined spectral density of the two shocks satisfies the Kolmogorov result[8].

Assumption 3 *:* $\sum_{j=1}^{t} \mathcal{D}(\omega_j, y_k, \boldsymbol{lNB}) \gg \sum_{j=t+1}^{T} \mathcal{D}(\omega_j, y_k, \boldsymbol{lNB})$

where $\sum_{j=1}^{t} \mathcal{D}(\omega_j, y_k, \mathbf{lNB}) \gg \sum_{j=t+1}^{T} \mathcal{D}(\omega_j, y_k, \mathbf{lNB})$ represents that a long-run Non-Business cycle shock would result in very low volatility for frequencies corresponding to short-run fluctuations i.e. $\sum_{j=t+1}^{T} \mathcal{D}(\omega_j, y_k, \mathbf{lNB})$ compared to frequencies corresponding to long-run fluctuations $\sum_{j=1}^{t} \mathcal{D}(\omega_j, y_k, \mathbf{lNB})$.

Lemma 1 *: The minimum[9] value of $S(\theta)$ when $\sigma^2_{\boldsymbol{lNB}} = 0$ is $T\frac{\sigma^2_{\boldsymbol{B}}}{2\pi}$.*

Theorem 1 *: Under Assumptions 1, 2 & 3, the minimization of $S(\theta)$ is achieved at true parameters θ^* if and only if $\sigma^2_{\boldsymbol{lNB}} = 0$.*

Proof: First, suppose $\sigma^2_{\mathbf{lNB}} = 0$,

From Assumption 1, θ^* is s.t. $\mathcal{D}(\omega_j, y_k, \mathbf{B}) = \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^*)\ \forall\ \omega_j$ where $j \in \{1, 2, \cdots, T\}$

Substituting $\mathcal{D}(\omega_j, y_k, \mathbf{B})$ with $\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^*)$ for all frequencies in the minimization problem (22)

$$S(\theta^*) = \quad T\frac{\sigma^2_{\mathbf{B}}}{2\pi} + \sum_{j=1}^{T} \frac{\mathcal{D}(\omega_j, y_k, \mathbf{lNB})\sigma^2_{\mathbf{lNB}}}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^*)}$$

Given $\sigma^2_{\mathbf{lNB}} = 0$, $\therefore S(\theta^*) = T\frac{\sigma^2_{\mathbf{B}}}{2\pi}$

Following Lemma 1, $S(\theta)$ is minimized at θ^* for $\sigma^2_{\mathbf{lNB}} = 0$.

Next, suppose $\sigma^2_{\mathbf{lNB}} > 0$,

Let $\sigma^2_{\mathbf{lNB}} = \kappa\ \sigma^2_{\mathbf{B}}$, substituting $\sigma^2_{\mathbf{lNB}}$ in (22)

[8] $\sum_{j=1}^{T} \log \frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})} = 0$

[9] Appendix 7.3 for minimum value analysis

$$S(\theta) = \frac{\sigma_{\mathbf{B}}^2}{2\pi}\left(\underbrace{\sum_{j=1}^{t}\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{long-run volatility}} + \underbrace{\sum_{j=t+1}^{T}\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{short-run volatility}}\right) \tag{24}$$

Divide and multiply (24) with $\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})$

where, $\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB}) = \prod_{j=1}^{T}\left(\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa\right)$

$$S(\theta) = \frac{\sigma_{\mathbf{B}}^2\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})}{2\pi}\left(\underbrace{\sum_{j=1}^{t}\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{long-run volatility}} + \underbrace{\sum_{j=t+1}^{T}\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{short-run volatility}}\right) \tag{25}$$

From Assumption 2, $\theta^{'}$ is s.t. $\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})} = \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{'})\ \forall\ \omega_j$ where $j \in \{1, 2, \cdots, T\}$

Substituting $\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})}$ with $\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{'})$ for all frequencies in the minimization problem (25)

$$S(\theta^{'}) = \quad T\frac{\sigma_{\mathbf{B}}^2\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})}{2\pi}$$

From AM-GM inequality, the minimum value of $S(\theta)$ in (25) is $T\frac{\sigma_{\mathbf{B}}^2\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})}{2\pi}$

Thus, $S(\theta)$ in the presence of both business and non-business cycle shocks is minimized at $\theta^{'}$.

From Assumption 3, $\sum_{j=1}^{t}\mathcal{D}(\omega_j, y_k, \mathbf{lNB}) \gg \sum_{j=t+1}^{T}\mathcal{D}(\omega_j, y_k, \mathbf{lNB})$

This implies the following inequality,

$$\frac{\sum_{j=1}^{t}\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\sum_{j=t+1}^{T}\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa} > \frac{\sum_{j=1}^{t}\mathcal{D}(\omega_j, y_k, \mathbf{B})}{\sum_{j=t+1}^{T}\mathcal{D}(\omega_j, y_k, \mathbf{B})}$$

Following Assumptions 1 & 2, substituting $\frac{\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa}{\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})}$ with $\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{'})$ & $\mathcal{D}(\omega_j, y_k, \mathbf{B})$ with $\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{*})$ for all frequencies in the above inequality

$$\frac{\sum_{j=1}^{t} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{'})}{\sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta')} > \frac{\sum_{j=1}^{t} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{*})}{\sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{*})} \tag{26}$$

Therefore, $\theta' \neq \theta^*$ □

Thus, under Assumptions 1, 2 & 3, the minimization of $S(\theta)$ is achieved at true parameters θ^* if and only if $\sigma^2_{\mathbf{lNB}} = 0$.

In the remainder of this section, I will underscore additional implications stemming from the aforementioned results.

Based on the result from (20), the estimated volatility of the business-cycle shock is equal to the true volatility when $\sigma^2_{\mathbf{lNB}} = 0$.

$$\tilde{\sigma}^2_{\mathbf{B}}(\theta^*) = \frac{2\pi}{T} S(\theta^*) = \sigma^2_{\mathbf{B}}$$

However, in the presence of significant long-run non-business cycle volatility $\left(\sigma^2_{\mathbf{lNB}} > 0\right)$, the estimated volatility of the business-cycle shock is upward biased as $\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB}) > 1$.

$$\tilde{\sigma}^2_{\mathbf{B}}(\theta^{'}) = \frac{2\pi}{T} S(\theta^{'}) = \sigma^2_{\mathbf{B}} \mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB})$$

The bias in the parameters would also result in a downward bias in model-implied business cycle implications when estimated in a full information setting with significant long-run non-business cycle fluctuations. Given the inequality (26) & cross-frequency restrictions in (23),

$$\sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{'}) < \sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{*}) = \sum_{j=t+1}^{T} \mathcal{D}(\omega_j, y_k, \mathbf{B})$$

This implies that the presence of long-run non-business cycle shocks would result in parameters such that the model-implied short-run volatility $\left(\sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta^{'})\right)$ would be lower than the actual data implied short-run volatility $\left(\sum_{j=t+1}^{T} \mathcal{D}(\omega_j, y_k, \mathbf{B})\right)$.

$$\sigma^2_{\mathbf{INB}} \uparrow \xRightarrow{minimize \quad S(\theta)} \underbrace{\sum_{j=1}^{t} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}_{\text{long-run}} \uparrow \xRightarrow{restriction} \underbrace{\sum_{j=t+1}^{T} \mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}_{\text{short-run}} \downarrow$$

For intuition, as the volatility ($\sigma^2_{\mathbf{INB}}$) of long-run non-business cycle shocks increases, the minimization of the objective function $S(\theta)$ results in a θ such that the model implied long-run volatility increases given the cross-frequency restrictions resulting in a downward bias on the short-run volatility of the model. Given the evidence of a significant fraction of long-run non-business cycle fluctuations from the previous section, this argues for estimation in a limited information setting via IRF matching with the identified business cycle shocks in section 3. The following section showcases the normative and policy implications of the SW07 by comparative analysis of the model, estimated using conditional moments, and the Bayesian likelihood estimation of the original estimated model.

5. Application: Smets & Wouter (2007)

This section centers on applying the above identification strategy for comparative analysis with full information estimation using a benchmark medium-scale DSGE model proposed by SW07. The model serves a two-fold purpose: firstly, to label the demand shock based on both the model and empirical impulse response functions (IRFs); secondly, to examine the normative and policy implications by identifying the model parameters through impulse response matching, a method commonly utilized, among others, in Christiano et al. (2005).

5.1 The Model

Their model includes monopolistic competition in goods and labor markets, sticky prices and wages, partial indexation of prices and wages, investment adjustment costs, habit persistence, and variable capacity utilization. The economy evolves along a balanced growth path, driven by deterministic labor-augmenting technological progress. The endogenous variables in the model, expressed as log-deviations from steady state, are output (y_t), consumption (c_t), investment (i_t), utilized and installed capital (k_t^s, k_t), capacity utilization (ϵ_t), rental rate of capital $\left(r_t^k\right)$, Tobin's $q\,(q_t)$, price and wage markup (μ_t^p, μ_t^w), inflation rate (π_t), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). The log-linearized equilibrium conditions for these variables are presented in the appendix (7.1). The last equation in the table gives the policy rule followed by the central bank, which sets the nominal interest rate in response to inflation and the deviation of output from its potential level. To determine potential output, defined as the level of output that would prevail without the price and wage mark-up shocks. The business cycle dynamics of the model are driven by seven stationary shocks. Five of them-total factor productivity (ε_t^a), investment-specific technology $\left(\varepsilon_t^i\right)$, government purchases (ε_t^g), risk premium $\left(\varepsilon_t^b\right)$, and monetary policy (ε_t^r)-follow AR(1) processes; the remaining two shocks-to wage and price markup $(\varepsilon_t^w$ and $(\varepsilon_t^p))$ - follow ARMA(1,1) processes.

5.2 Identification analysis

The model is estimated using data on seven variables: real GDP, real consumption, real investment, real wage, inflation, hours worked and the nominal interest rate. Thus, the vector of observables is given by

$$\boldsymbol{x}_t = [y_t, c_t, inv_t, w_t, \pi_t, l_t, r_t,]' \tag{27}$$

For comparative analysis with the dissection strategy suggested in this paper, the SW07 model is estimated with Bayesian likelihood estimation techniques in SW07 using seven key macroeconomic quarterly US time series as observable variables: the log difference of real GDP, real consumption, real investment, and the real wage, log hours worked, the log difference of the GDP deflator, and the federal funds rate. Then, the business cycle shock identification strategy is applied to a VAR with the same seven observables.

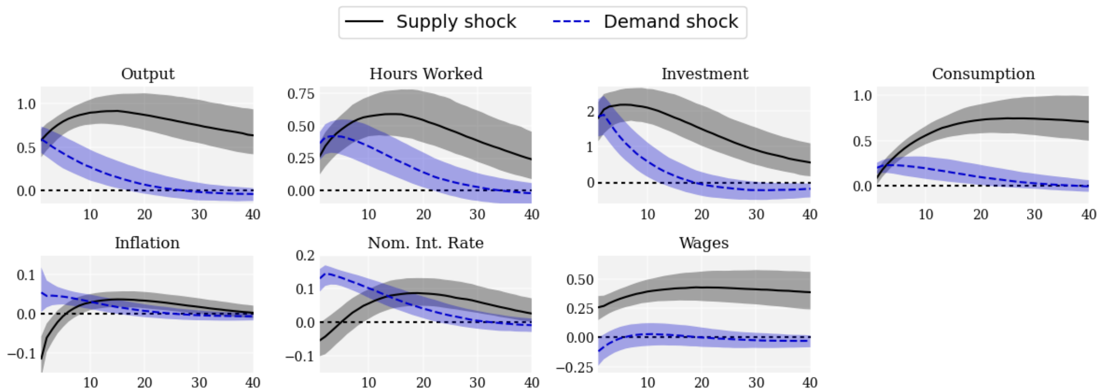
The impulse response results, reflecting both long-run and short-run shocks, consistently align with the identification presented in the benchmark VAR model introduced by ACD in the preceding section. The identification process is substantiated by the negative conditional correlation observed between real GDP and inflation, the long-run persistence of the real GDP impulse response as depicted in Figure 4, and the substantial explanatory power exhibited by the macroeconomic variables' business cycle and long-run volatility, as demonstrated in Table 3. This alignment highlights the identification of the long-run shock being attributed to a supply shock.

Similarly, the identification of the short-run shock as a demand shock is reinforced by the positive conditional correlation observed between real GDP, inflation, and the federal funds rate. These conditional correlations are evident in Figure 4. Moreover, the explanation of the identified model to explain the business-cycle volatility across various macro-variables, highlighted in Table 3, further substantiates the identification of the short-run shock as a demand shock.

Given the array of seven shocks in SW07, there are only two shocks with potential to genrate characterstic business-cycle comovement. Within the category of demand shocks that comprises, discount factor shock, risk-premia shock, monetary shock, investment shock, spending shock, and monetary shock it is the risk premium shock helps to explain the comovement of consumption and investment in presence of nominal rigidities. This encourages the further identification of the demand shock as a risk premia shock.

In the upcoming subsection, I delve into the identification of parameters by employing impulse response matching with the same set of observables utilized in SW07. This involves using the previously identified long-run GDP shock as a supply shock and risk-premia as demand business cycle shocks.

Figure 4: Smets & Wouter (2007) VAR: Impulse Response Functions



Impulse Response Functions of all the variables to the identified supply shock. Horizontal axis: time horizon in quarters. Shaded area: 68 percent Highest Posterior Density Interval (HPDI).

Table 3: Variance Contributions

	Y	h	I	C
Supply Shock: Short run (6–32 quarters)	57.3	23.8	42.9	26.7
	[33.9, 76]	[10.5, 36.1]	[22.7, 60.3]	[18.7, 36.6]
Supply Shock: Long run (80–400 quarters)	66.5	69.8	69.1	65
	[36.9, 86.4]	[44.8, 84.4]	[40, 86.5]	[34.9, 85.3]
Demand Shock: Short run (6–32 quarters)	42.5	30.7	38.4	17.9
	[23.6, 65.7]	[16, 45.6]	[20.4, 59.8]	[8.3, 27.8]
	FFR	**w**	**Δp**	
Supply Shock: Short run (6–32 quarters)	15.31	19.78	22.2	
	[6.3, 30.9]	[10.5, 32.5]	[10.2, 37.9]	
Supply Shock: Long run (80–400 quarters)	27.7	65.3	27.7	
	[12.9, 51.7]	[33.8, 85.2]	[11, 56]	
Demand Shock: Short run (6–32 quarters)	41.2	5.5	8.3	
	[25.4, 53.6]	[2.3, 12.7]	[3.1, 20.9]	

Notes: Variance contributions of the MBC shock at two frequency bands. The first row (Short run) corresponds to the range between 6 and 32 quarters, and the second row (Long run) to the range between 80 quarters and ∞. The notation used for the variables is the same as that introduced in Section I. 68 percent HPDI in brackets.

5.3 Estimated Parameters

Consider a model that matches impulse responses of all seven variables to two structural shocks, the identified long-run TFP and a risk-premia shock. Following rank conditions of Iskrev (2010) & Komunjer and Ng (2011), two of the unidentifiable parameters are $\bar{l}, \bar{\pi}$, which affect only the mean of observables; in addition, there is a set of five parameters, namely $\delta, \beta, \Theta, \lambda$ and γ, any four of which can be identified only if the fifth one is known. Also, the lack of identification corresponding to ε_p and ξ_p on one hand, and ε_w and ξ_w, on the other. As in SW07, I assume that the curvature parameters ε_w, and ε_p are known and are both equal to 10. Following SW07, I assume the calibrated values of capital depreciation rate δ, steady state wage markups and fixed cost chare. Additionaly, the parameters that characterize the stochastic properties of the excluded structural shocks are also unidentifiable. A TFP shock activates the government purchases process, thus identifying ρ_{ga} and ρ_g (see Eq. (21) 7.1) in addition to 21 other parameters. Inclusion of risk premia shocks extends the number of parameters estimated to 23. Table **??** reports the estimated parameter values.

5.4 Comparative Analysis of Estimation Methods

Figure 5 presents impulse response functions (IRFs) to illustrate the effects of different estimation methods on a model's performance. The blue lines depict the IRFs of the model estimated through IRF matching with identified business cycle shocks (shown in red) in response to a one-standard-deviation Total Factor Productivity (TFP) shock and a risk-premia shock. This estimation method accurately captures the dynamic responses of these identified shocks. Additionally, I include the IRFs of the SW07 model estimated using Bayesian likelihood, represented in black. While the Bayesian estimation aligns well with the empirical IRFs in most aspects, there are disparities in investment and hours worked.

To comprehend the implications of these two estimation methods, this section illustrates the comparative analysis in two parts.

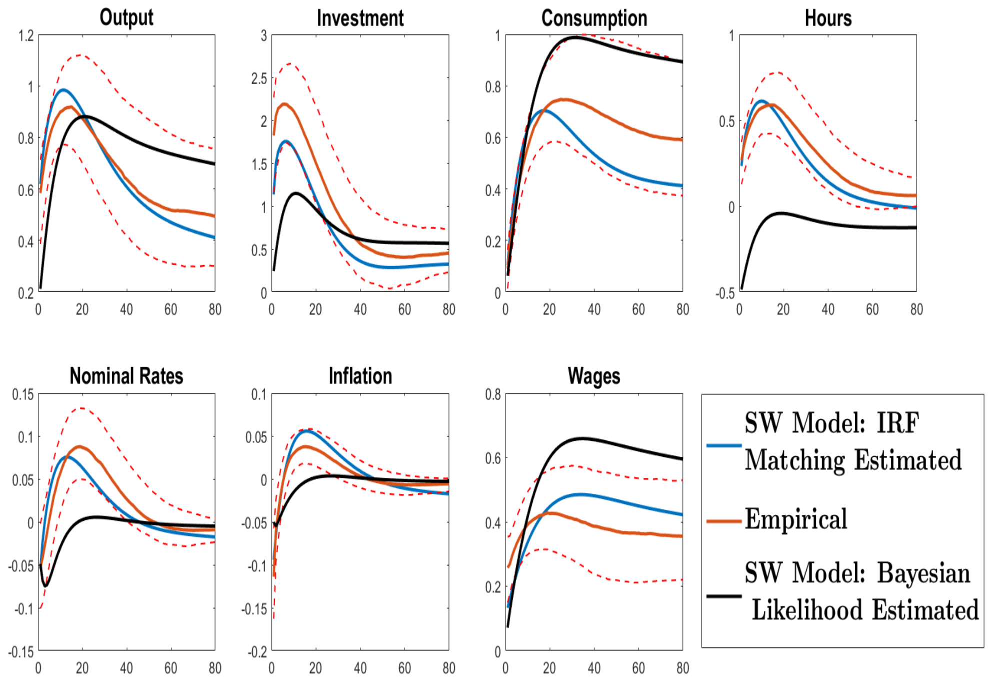
Firstly, Figure 7 focuses on the response to a long-run TFP shock. Here, the two estimation methods lead to significantly different normative and policy implications. In the Bayesian estimated model (in black), inflation and the output gap exhibit a positive correlation, resembling the divine coincidence often seen in New Keynesian models with demand shocks. In contrast, the model estimated via IRF matching with identified business cycle shocks reveals a negative conditional correlation between the output gap and inflation. This suggests a policy tradeoff for the monetary authority: lowering interest rates to counter falling inflation may lead to a further increase in the output gap.

Secondly, through Figure 8, I argue the two estimations result in different inferences about the underlying internal and external propagation mechanisms of SW07. The mentioned figure has empirical and model IRFs for a risk premia shock. The standard deviation of the risk premia shock (σ_{rp}) for the Bayesian likelihood estimated is 0.1762. I first replaced it with the IRF matching estimated σ_{rp} of 0.0131 which is 17 times smaller than the full information setting value while I kept the rest of the parameters estimated by Bayesian likelihood estimation unchanged. As one may observe in the black IRFs, in the same figure, the risk premia shock becomes insignificant. Next I replace the investment elasticity estimated by Bayesian likelihood (8.0145) with the IRF matching estimated value of 0.0145 and as we may observe risk premia IRFs become significant on impact and persistence. This showcases the key differences in estimation where full information setting argues for a stronger external propagation role in risk premia-driven business cycles while IRF matching argues for a stronger internal propagation through financial frictions.

In summary, the two estimations result in key differences for the same model over the same dataset in terms of normative and policy implications which is crucial given the use of augmented versions of such medium-scale DSGE models at central banks. Based on theoretical insights from the previous section, it argues for limited information estimation to avoid any biased business cycle implications because of non-business cycle information in the full information settings.

Figure 5: Smets & Wouters (2007): Impulse Response Functions

(a) Long-run TFP Shock



(b) Risk Premia Shock

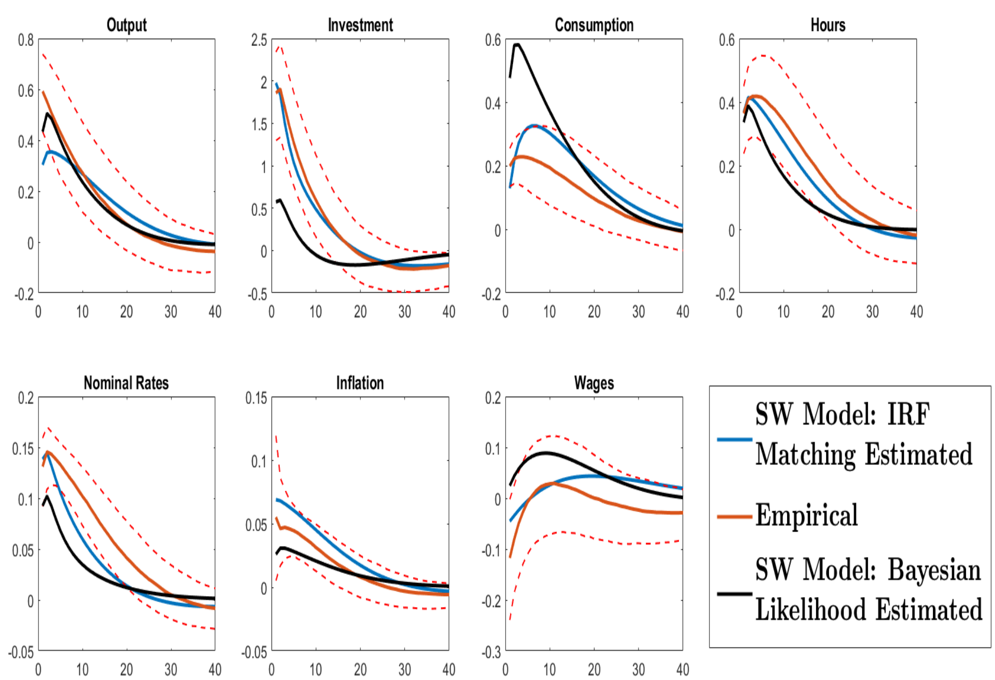


Figure 7: Long-run TFP Shock: Output Gap & Inflation

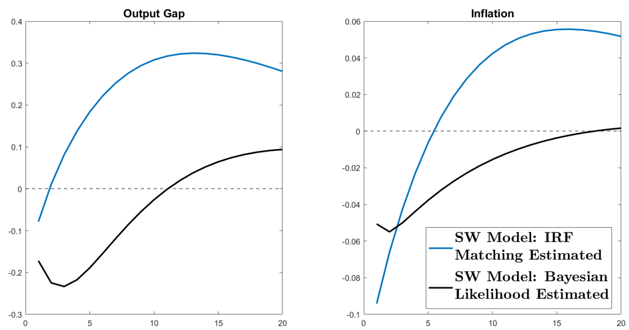
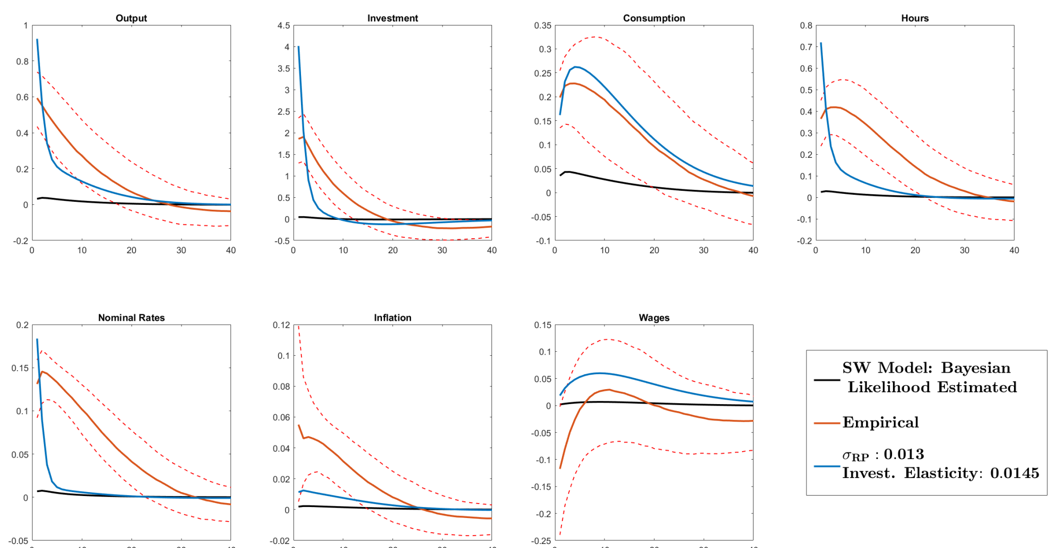


Figure 8: Risk Premia Shock: Internal vs. External Propagation



6. Conclusion

The paper introduces a novel approach for analyzing GDP fluctuations, uncovering the dynamic causal effects of long-run supply and short-run demand shocks as drivers of business cycles. These shocks collectively account for nearly 99% of GDP's business cycle volatility and over 50% for other key macroeconomic variables.

Furthermore, the empirical findings reveal the existence of a second category of long-run supply shocks that do not influence GDP's business cycle fluctuations. This, coupled with theoretical insights, emphasizes the parameter bias inherent in medium-scale DSGE models estimated under full information settings. These biased parameters can lead to a downward distortion of the business cycle implications derived from estimated DSGE models.

By estimating medium-scale DSGE models using conditional moments derived from identified business cycle shocks, the paper compares the normative and policy implications of the same model when estimated under both full information and limited information settings.

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7. Appendix

Table 4: Estimated Parameter Values

Parameter	Description	IRF Matching	Bayesian Likelihood
ρ_{ga}	Feedback technology on exogenous spending	0.9905	0.2272
$100\left(\beta^{-1}-1\right)$	time preference rate in percent	1.7162	0.1239
α	capital share	0.178	0.2079
ψ	capacity utilization cost	0.9658	0.6723
Θ	investment adjustment cost	0.0145	8.0415
σ_c	risk aversion	1.5866	1.3295
λ	external habit degree	0.6084	0.8789
Θ	fixed cost share	1	1.4888
l_w	Indexation to past wages	0.8241	0.5542
ξ_w	Calvo parameter wages	0.86	0.8682
l_p	Indexation to past prices	0	0.2127
ξ_p	Calvo parameter prices	0.72	0.7697
σ_l	Frisch elasticity	0.25	2.2934
r_π	Taylor rule inflation feedback	1	1.7822
$r_{\triangle y}$	Taylor rule output growth feedback	0.3835	0.0010
r_y	Taylor rule output level feedback	0.064	0.1907
ρ	interest rate persistence	0.7522	0.8283
ρ_a	persistence productivity shock	0.9974	0.9975
ρ_b	persistence risk premium shock	0.83	0.2751
ρ_g	persistence spending shock	0.9795	0.9810
γ	growth rate	1	1.0032
σ_a	Std. productivity shock	0.4247	0.5557
σ_b	Std. risk premium shock	0.0131	0.1762

7.1 Log-linearized equations of the SW07 model (sticky-price-wage economy)

$$
\begin{aligned}
y_t &= c_y c_t + i_y i_t + r^{kss} k_y \epsilon_t + \varepsilon_t^g \\
c_t &= \frac{\lambda/\gamma}{1+\lambda/\gamma} c_{t-1} + \frac{1}{1+\lambda/\gamma} \mathrm{E}_t c_{t+1} + \frac{w^{ss} s^{ss} (\sigma_c - 1)}{c^{ss} \sigma_c (1+\lambda/\gamma)} (l_t - \mathrm{E}_\ell l_{t+1}) \\
&- \frac{1-\lambda/\gamma}{(1+\lambda/\gamma)\sigma_c} (r_t - \mathrm{E}_t \pi_{t+1}) - \frac{1-\lambda/\gamma}{(1+\lambda/\gamma)\sigma_c} e_t^b
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
i_l &= \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} i_{l-1} + \frac{\beta\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} \mathrm{E}_t i_{\ell+1} + \frac{1}{\Theta\gamma^2 (1+\beta\gamma^{(1-\sigma_c)})} q_t + \varepsilon_t^i \\
q_t &= \beta(1-\delta)\gamma^{-\sigma_c} \mathrm{E}_t q_{t+1} - r_t + \mathrm{E}_t \pi_{t+1} + (1 - \beta(1-\delta)\gamma^{-\sigma_c}) \mathrm{E}_t r_{t+1}^k - \varepsilon_t^b \\
y_t &= \Theta_p (\alpha k_t^s + (1-\alpha) l_t + \varepsilon_t^a) \\
k_l^s &= k_{t-1} + \epsilon_t \\
\epsilon_t &= \frac{1-\psi}{\psi} r_t^k \\
k_t &= (1-\delta)/\gamma k_{t-1} + (1 - (1-\delta)/\gamma) i_t + (1 - (1-\delta)/\gamma) \varphi \gamma^2 (1+\beta\gamma^{(1-\sigma_c)}) \varepsilon_l^i \\
\mu_t^p &= \alpha (k_t^s - l_t) - w_t + \varepsilon_t^a \\
\pi_t &= \frac{\beta\gamma^{(1-\sigma_c)}}{1 + I_p \beta\gamma^{(1-\sigma_c)}} \mathrm{E}_t \pi_{t+1} + \frac{I_p}{1+\beta\gamma^{1-\sigma_c I_p}} \pi_{t-1} \\
&- \frac{(1-\beta\gamma^{(1-\sigma_c)}\xi_p)(1-\xi_p)}{(1+I_p\beta\gamma^{(1-\sigma_c)})(1+(\Theta_p - 1)\varepsilon_p)\xi_p} \mu_t^p + \varepsilon_t^p \\
r_t^k &= l_t + w_t - k_t \\
\mu_t^w &= w_t - \sigma_l l_t - \frac{1}{1-\lambda/\gamma} (c_t - \lambda/\gamma c_{t-1}) \\
w_t &= \frac{\beta\gamma^{(1-\sigma_c)}}{1+\beta\gamma^{(1-\sigma_c)}} (\mathrm{E}_t w_{t+1} + \mathrm{E}_t \pi_{t+1}) + \frac{1}{1+\beta\gamma^{(1-\sigma_c)}} (w_{t-1} + l_w \pi_{t-1}) \\
&- \frac{1+\beta\gamma^{(1-\sigma_c)} I_w}{1+\beta\gamma^{(1-\sigma_c)}} \pi_t - \frac{(1-\beta\gamma^{(1-\sigma_c)}\xi_w)(1-\xi_w)}{(1+\beta\gamma^{(1-\sigma_c)})(1+(\varphi_w - 1)\varepsilon_w)\xi_w} \mu_t^w + \varepsilon_t^w \\
r_t &= \rho r_{t-1} + (1-\rho)(r_\pi \pi_t + r_y (y_t - y_t^*)) \\
&+ r_{\Delta y} ((y_t - y_t^*) - (y_{t-1} - y_{t-1}^*)) + \varepsilon_t^r \\
\varepsilon_q^a &= \rho_a \varepsilon_{t-1}^a + \eta_t^a \\
\varepsilon_t^b &= \rho_a \varepsilon_{t-1}^b + \eta_t^b \\
\varepsilon_\tau^g &= \rho_g \varepsilon_{t-1}^a + \rho_{ga} \eta_t^a + \eta_t^g
\end{aligned} \tag{29}
$$

Note: The model variables are: output (y_t), consumption (c_t), investment (i_t), utilized and installed capital (k_t^s, k_t), capacity utilization (ϵ_t), rental rate of capital $\left(r_t^k\right)$, Tobin's q (q_t), price and wage markup (μ_t^p, μ_t^w), inflation rate (π_t), real wage (w_t), total hours worked (l_t), and nominal interest rate (r_t). The shocks are: total factor productivity (ε_t^a), investment-specific technology $\left(\varepsilon_t^i\right)$, government purchases (ε_t^g), risk premium $\left(\varepsilon_t^b\right)$, monetary policy (ε_t^r), wage markup (ε_t^w) and price markup (ε_t^p).

7.2 MBC of ACD

Consider our simple bivariate VAR

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{\text{supply}} \\ \varepsilon_t^{\text{demand}} \end{bmatrix} \tag{30}$$

The Impulse response can be computed recursively as follows

$$\begin{cases} IR_t = Bs & \text{for } t = 0 \\ IR_t = \Phi IR_{t-1} & \text{for } t = 1, \ldots, h \end{cases} \tag{31}$$

$$\begin{bmatrix} IR_0^y \\ IR_0^r \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} \tag{32}$$

$$\begin{cases} \text{VD}_{y_0}^{\varepsilon^{\text{supply}}} & = \frac{b_{11}^2}{b_{11}^2 + b_{12}^2} \\ \text{VD}_{y_0}^{\varepsilon^{\text{demand}}} & = \frac{b_{12}^2}{b_{11}^2 + b_{12}^2} \end{cases} \quad \begin{cases} \text{VD}_{r_0}^{\varepsilon^{\text{supply}}} & = \frac{b_{21}^2}{b_{21}^2 + b_{22}^2} \\ \text{VD}_{r_0}^{\varepsilon^{\text{demand}}} & = \frac{b_{22}^2}{b_{21}^2 + b_{22}^2} \end{cases} \tag{33}$$

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} q_{11} & q_{21} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{\text{supply}} \\ \varepsilon_t^{\text{demand}} \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} y_t \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} q_{11}\varepsilon_t^{\text{supply}} + q_{21}\varepsilon_t^{\text{demand}} \\ q_{12}\varepsilon_t^{\text{supply}} + q_{22}\varepsilon_t^{\text{demand}} \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} y_t \\ \\ r_t \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ \\ r_{t-1} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{\text{MBC}} \\ \\ \varepsilon_t^{\text{SBC}} \end{bmatrix} \tag{36}$$

$$\begin{cases} \text{VD}_{y_0}^{\varepsilon^{\text{MBC}}} & = \frac{q_{11}^2 b_{11}^2 + q_{21}^2 b_{12}^2 + 2q_{11}q_{21}b_{11}b_{12}}{b_{11}^2 + b_{12}^2} \\ \text{VD}_{y_0}^{\varepsilon^{\text{SBC}}} & = \frac{q_{12}^2 b_{11}^2 + q_{22}^2 b_{12}^2 + 2q_{12}q_{22}b_{11}b_{12}}{b_{11}^2 + b_{12}^2} \end{cases} \qquad \begin{cases} \text{VD}_{r_0}^{\varepsilon^{\text{MBC}}} & = \frac{q_{11}^2 b_{21}^2 + q_{21}^2 b_{22}^2 + 2q_{11}q_{21}b_{21}b_{22}}{b_{21}^2 + b_{22}^2} \\ \text{VD}_{r_0}^{\varepsilon^{\text{SBC}}} & = \frac{q_{12}^2 b_{21}^2 + q_{22}^2 b_{22}^2 + 2q_{12}q_{22}b_{21}b_{22}}{b_{21}^2 + b_{22}^2} \end{cases} \tag{37}$$

Suppose $b_{11} > 0$ & $b_{12} > 0$. Therefore, $b_{21} < 0$ & $b_{22} > 0$

Selecting q_{11} & q_{21} such that $\text{VD}_{y_0}^{\varepsilon^{\text{MBC}}}$ is maximized. This implies $q_{11} > 0$ & $q_{21} > 0$

Resulting in fall of $\text{VD}_{r_0}^{\varepsilon^{\text{MBC}}}$ as $b_{21}b_{22} < 0$

7.3 Minimum Value of the Objective Function

$$S(\theta) = \underbrace{\sum_{j=1}^{t} \frac{1}{2\pi} \frac{\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma_{\mathbf{B}}^2 + \mathcal{D}(\omega_j, y_k, \mathbf{INB})\sigma_{\mathbf{INB}}^2}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{long-run volatility}} + \underbrace{\sum_{j=t+1}^{T} \frac{1}{2\pi} \frac{\mathcal{D}(\omega_j, y_k, \mathbf{B})\sigma_{\mathbf{B}}^2 + \mathcal{D}(\omega_j, y_k, \mathbf{INB})\sigma_{\mathbf{INB}}^2}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)}}_{\text{short-run volatility}} \tag{38}$$

Let

$$x_j = \frac{\mathcal{D}(\omega_j, y_k, \mathbf{B})}{\mathcal{M}(\omega_j, y_k, \mathbf{B}; \theta)} \tag{39}$$

when $\sigma_{\mathbf{INB}}^2 = 0$

$$S(\theta) = \frac{\sigma_{\mathbf{B}}^2}{2\pi} \sum_{j=1}^{T} x_j \tag{40}$$

AM-GM Inequality: For any list of T nonnegative real numbers $x_1, x_2, \ldots, x_T$,

$$\frac{x_1 + x_2 + \cdots + x_T}{T} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_T} \tag{41}$$

and that equality holds if and only if $x_1 = x_2 = \cdots = x_T$.

Therefore,

$$S(\theta) \geq \frac{T\sigma_{\mathbf{B}}^2}{2\pi} \sqrt[n]{x_1 \cdot x_2 \cdots x_T} \tag{42}$$

$$S(\theta) = \frac{T\sigma_{\mathbf{B}}^2}{2\pi} \tag{43}$$

iff $x_1 = x_2 = \cdots = x_T = 1$

7.4 Multiple Business Cycle Shocks

The following subsection shows that multiple shocks can be mapped to a single business cycle shock.

Using the state-space representation and mapping to a vector of observables $\mathbf{Y}_t$:

$$S_t = \Theta_1(\theta)S_{t-1} + \Theta_\epsilon(\theta)\Psi(\theta_1)\epsilon_t$$

$$Y_t = A(L)S_t = A(L)\left(I - \Theta_1(\theta)L\right)^{-1}\Theta_\epsilon(\theta)\Psi(\theta_1)\epsilon_t = \mathbf{D}(L;\theta)\mathbf{\Theta}_\epsilon(\theta)\mathbf{\Psi}(\theta_1)\epsilon_t$$

$$Y_t(k) = \mathbf{D}(L;\theta)\mathbf{\Theta}_\epsilon(\theta)\mathbf{\Psi}(\theta_1)\epsilon_t$$

$$Y_t(k, sB, lB) = \mathbf{D^k}(L;\theta)\mathbf{\Theta}_\epsilon^{sB}\sigma_{sB}\epsilon_t^{sB} + \mathbf{D^k}(L;\theta)\mathbf{\Theta}_\epsilon^{lB}\sigma_{lB}\epsilon_t^{lB}$$

Let $\sigma_{\mathbf{sB}} = \gamma\sigma_{\mathbf{lB}}$

$$Y_t(k, sB, lB) = \left(\mathbf{D^k}(L;\theta)\mathbf{\Theta}_\epsilon^{sB}\gamma + \mathbf{D^k}(L;\theta)\mathbf{\Theta}_\epsilon^{lB}\right)\sigma_{lB}\epsilon_t^{B}$$

$$Y_t(k, sB, lB) = \mathbf{D^k}(L;\theta)\left(\mathbf{\Theta}_\epsilon^{sB}\gamma + \mathbf{\Theta}_\epsilon^{lB}\right)\sigma_{lB}\epsilon_t^{B}$$

Let $\Theta_\epsilon^B\sigma_B = \left(\mathbf{\Theta}_\epsilon^{sB}\gamma + \mathbf{\Theta}_\epsilon^{lB}\right)\sigma_{lB}$

The model implied Spectral Density of variable **k** in Y_t due to business cycle shock $\left(\varepsilon_t^B\right)$ at frequency ω is represented as:

$$\mathcal{SD}(\omega, k, \mathbf{B}; \theta, \theta_1) = \frac{1}{2\pi}\mathcal{M}(\omega, y_k, \mathbf{B}; \theta)\sigma_{\mathbf{B}}^2, \quad \text{where} \quad \mathcal{M}(\omega, y_k, \mathbf{B}; \theta) = \left|\mathbf{D}^k(e^{i\omega};\theta)\mathbf{\Theta}_\epsilon^{\mathbf{B}}(\theta)\right|^2$$

7.5 Upward Bias

$$\begin{aligned}\mathcal{D}(y_k, \mathbf{B}, \mathbf{lNB}) &= \prod_{j=1}^{T}\left(\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa\right) \\ &\geq \prod_{j=1}^{T}\mathcal{D}(\omega_j, y_k, \mathbf{B}) + \prod_{j=1}^{T}\mathcal{D}(\omega_j, y_k, \mathbf{lNB})\kappa \quad = 1 + \kappa^T\end{aligned}$$

where for $\sigma^2_{\mathbf{INB}} > 0$,

$$\kappa = \frac{\sigma^2_{\mathbf{INB}}}{\sigma^2_{\mathbf{B}}} > 0$$